



TITLE:

# On the Hausdorff dimension of the attractor for the heat convection equation(Mathematical Analysis of Phenomena in Fluid and Plasma Dynamics)

AUTHOR(S):

OEDA, Kazuo

---

CITATION:

OEDA, Kazuo, On the Hausdorff dimension of the attractor for the heat convection equation(Mathematical Analysis of Phenomena in Fluid and Plasma Dynamics). 数理解析研究所講究録 1993, 824: 212-223

ISSUE DATE:

1993-03

URL:

<http://hdl.handle.net/2433/83240>

RIGHT:

# On the Hausdorff dimension of the attractor for the heat convection equation

By Kazuo ŌEDA (Japan Women's University)

大枝 一男 (日本女子大学)

## §1. Introduction.

In our previous study [3], we considered the heat convection equation (HC) in a time-dependent domain  $\Omega(t) \subset \mathbb{R}^2$  and showed the existence of the absorbing set for (HC).

On the other hand, Foias-Manley-Temam [1] showed the existence of the attractor for the Bénard problem and obtained the estimates of the Hausdorff and fractal dimensions of the attractor.

In this paper we consider (HC) in a wider class of fixed bounded domains of  $\mathbb{R}^2$  with inhomogeneous boundary conditions and we estimate the Hausdorff and fractal dimensions of the attractors for (HC).

## §2. Equations and assumptions.

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  included in an open ball  $B = B(0, d)$ . The boundary  $\partial\Omega$  consists of  $N$  connected components, namely,  $\partial\Omega = \Gamma_1 + \cdots + \Gamma_N$ , where  $\Gamma_i$  are smooth (say, of class  $C^2$ ) and they does not intersect each other.

We consider the following heat convection equation :

$$(1) \quad \begin{cases} u_t + (u \cdot \nabla)u = -\nabla p / \rho + \{1 - \alpha(\theta - T_0)\}g + \nu \Delta u, \\ \operatorname{div} u = 0, \\ \theta_t + (u \cdot \nabla)\theta = \kappa \Delta \theta \quad \text{in } \Omega, \end{cases}$$

$$(2) \quad u|_{\partial\Omega} = \beta(x), \quad \theta|_{\partial\Omega} = T(x) \geq 0,$$

$$(3) \quad u|_{t=0} = a(x), \quad u|_{t=0} = h(x), \quad x \in \Omega,$$

where  $u$ ,  $p$  and  $\theta$  denote the velocity of the fluid, the pressure and the temperature, respectively;  $g(x)$  means the gravitational vector and  $\nu$ ,  $\kappa$ ,  $\alpha$ ,  $\beta$  are physical constants.

Now we make an assumption on the boundary function  $\beta$ .

Assumption.  $\beta$  is smooth and satisfies the condition

$$(4) \quad \int_{\Gamma_k} \beta \cdot n \, ds = 0 \quad (k = 1, \dots, N),$$

where  $n$  is the outer normal vector to  $\Gamma_k$ .

Then, the next lemma is known:

Lemma 1. Let  $\beta \in H^{3/2}(\partial\Omega)$ , then for any  $\varepsilon > 0$ , there exists  $b \in H^2(\Omega)$  such that  $b = \beta$  on  $\partial\Omega$ ,  $\operatorname{div} b = 0$  and

$$(5) \quad |((\nabla \cdot \nabla)b, v)| \leq \varepsilon \|\nabla v\|^2 \quad \text{for any } v \in H^1_0(\Omega).$$

Remark 1. We assume the function  $T(x)$  is continuous on  $\partial\Omega$ . Then we can have a function  $\bar{\theta}(x)$  such that  $\Delta \bar{\theta} = 0$  in  $\Omega$  and  $\bar{\theta}(\partial\Omega) = T(x)$ .

Now we make changes of variables:  $u = \hat{u} + b$ ,  $\theta = \hat{\theta} + \bar{\theta}$ ;  
 $(x, y) = d(x^*, y^*)$ ,  $t = (d^2/\nu) \cdot t^*$ ,  $\hat{u} = (\nu/d)u^*$ ,  $\hat{\theta} = (\nu T_0/\kappa)\theta^*$   
 and  $p = (\rho\nu^2/d^2)p^*$ , where  $T_0 = \max_x T(x)$ . Abbreviating asterisks  
 \* and using the same letters  $u$ ,  $\theta$ ,  $p$ ,  $x$ ,  $y$ ,  $t$ , the heat  
 convection equation (1) ~ (3) are rewritten as follows:

$$(6) \quad \begin{cases} u_t + (u \cdot \nabla)u = -\nabla p + \Delta u - (u \cdot \nabla)b - (b \cdot \nabla)u - R\theta - (b \cdot \nabla)b + \Delta b \\ \quad + d^3 g/\nu^2 - R(\bar{\theta} - 1/P), \\ \operatorname{div} u = 0, \end{cases}$$

$$\begin{aligned} & \theta_t + (u \cdot \nabla) \theta = (1/P) \Delta \theta - (u \cdot \nabla) \bar{\theta} - (b \cdot \nabla) \theta - (b \cdot \nabla) \bar{\theta} \quad , \\ (7) \quad & u|_{\partial\Omega} = 0 \quad , \quad \theta|_{\partial\Omega} = 0 \quad , \end{aligned}$$

$$(8) \quad u|_{t=0} = a-b \quad , \quad \theta|_{t=0} = h-\bar{\theta} \quad ,$$

where  $R = \alpha g T_0 d^3 / k\nu$  and  $P = \nu / \kappa$ .

We introduce the following abstract heat convection equation (AHC) :

$$(AHC) \quad \frac{dU}{dt} + AU(t) + FU(t) + MU(t) = P(\Omega)f,$$

here  $U = {}^t(u, \theta)$ ,  $AU(t) = {}^t(-P_\sigma(\Omega)(\Delta u), -(1/P)\Delta\theta)$ ,  $FU(t) =$

${}^t(P_\sigma(\Omega)((u \cdot \nabla)u, (u \cdot \nabla)\theta)$ ,  $MU(t) = {}^t(P_\sigma(\Omega)((u \cdot \nabla)b + (b \cdot \nabla)u + R\theta),$

$(u \cdot \nabla)\bar{\theta} + (b \cdot \nabla)\theta)$ ,  $f = {}^t(-(b \cdot \nabla)b + \Delta b + d^3 g / \nu^2 - R(\bar{\theta} - 1/P),$

$-(b \cdot \nabla)\bar{\theta})$ ,  $P(\Omega) = {}^t(P_\sigma(\Omega), 1_\Omega)$  and  $P_\sigma(\Omega)$  is the projection  $L^2(\Omega) \rightarrow H_\sigma(\Omega)$ .

### §3. Results.

To explain our results, we give some preliminaries.

Definition 1. Let  $U : [0, T] \rightarrow H_\sigma(\Omega) \times L^2(\Omega)$ ,  $T \in (0, \infty)$ .

Then  $U$  is called a strong solution of (AHC) on  $[0, T]$  if it satisfies the following properties (i) and (ii).

(i)  $U \in C([0, T] ; H_\sigma(\Omega) \times L^2(\Omega))$  and  $U(t)$  is absolutely continuous on  $(0, T]$ .

(ii)  $U(t) \in D(A) = (H^2(\Omega) \cap H_\sigma^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$  for a.e.  $t \in [0, T]$  and  $U$  satisfies (AHC) for a.e.  $t \in [0, T]$ .

Definition 2. If a strong solution  $U$  of (AHC) satisfies

$$(9) \quad U(0) = U_0 = {}^t(a-b, h-\bar{\theta}) \quad \text{in } H_\sigma(\Omega) \times L^2(\Omega),$$

then it is called a strong solution of the initial value problem for (AHC).

Here we put  $H = H_\sigma(\Omega) \times L^2(\Omega)$  and  $V = H_\sigma^1(\Omega) \times H_0^1(\Omega)$ .

Then we have the following existence theorem ([4]).

**Theorem 0.** Suppose the assumptions hold. Then for any  $U_0 \in H$  there exists a unique strong solution  $U$  with  $U(0) = U_0$  such that  $U \in C([0, T]; H) \cap L^2(0, T; V)$  and  $dU/dt \in L^2(\delta, T; H)$  where  $\delta$  is an arbitrary number in  $(0, T)$ . In particular, if  $U_0 \in V$ , then  $U \in C([0, T]; V) \cap L^2(0, T; D(A))$  and  $dU/dt \in L^2(0, T; H)$ .

Put  $S(t) : U_0 \rightarrow U(t)$ ,  $U(t)$  being a solution, then we have

**Theorem 1.** There exists a  $V$ -bounded absorbing set  $A$  in  $V$  for (AHC) in the following sense : For every bounded set  $B \subset V$ , there exists  $t = t(B) > 0$  such that  $S(t)B \subset A$  for all  $t \geq t(B)$ . Furthermore, for any bounded set  $B' \subset H$  we can take  $t(B') > 0$  satisfying  $S(t)B' \subset A$  for all  $t \geq t(B')$ .

Next we state the definition of an attractor.

**Definition 3.** Let  $\{S(t)\}_{t \geq 0}$  be a semigroup of continuous operators in a Hilbert space  $H$ . Then a functional invariant set for  $S(t)$  is a set  $X \subset H$  such that  $S(t)X = X$  for any  $t > 0$ .

**Definition 4.** Let  $X$  be a functional invariant set for  $S(t)$ . Then  $X$  is said to be an attractor in  $H$  if it possesses a neighbourhood  $\mathcal{O}$  of  $X$  in  $H$  such that for any  $\varphi_0 \in \mathcal{O}$   $\text{dist}(S(t)\varphi_0, X) \rightarrow 0$  as  $t \rightarrow \infty$ .

Then we have

Theorem 2. Let  $A$  be the absorbing set obtained in Theorem 1.

Putting  $X = \bigcap_{s \geq 0} \bigcup_{t \geq s} \overline{S(t)A}^H$ , then  $X$  is an attractor for (AHC).

Here we introduce the Hausdorff dimension of  $X$ .

Definition 5. Let  $E$  be a metric space and  $X$  be a subset of  $E$ . The number  $\mu_H(X, d) \in [0, \infty]$  defined by

$$\mu_H(X, d) = \lim_{\varepsilon \rightarrow 0} \mu_H(X, d, \varepsilon) = \sup_{\varepsilon > 0} \mu_H(X, d, \varepsilon)$$

is the  $d$ -dimensional Hausdorff measure of  $X$ , where

$$\mu_H(X, d, \varepsilon) = \inf \sum_i r_i^d$$

and the infimum is for all covering of  $X$  by a family  $(B_i)$  of balls of  $E$  of radii  $r_i \leq \varepsilon$ .

Definition 6. Let  $E$  be a metric space and  $X$  be a subset of  $E$ . The number  $d_H(X) \in [0, \infty]$  is called the Hausdorff dimension of  $X$  if it satisfies

$$\mu_H(X, d) = \begin{cases} 0 & , \quad d > d_H(X) \\ +\infty & , \quad d < d_H(X) \end{cases} ,$$

where  $\mu_H(X, d)$  is the  $d$ -dimensional Hausdorff measure of  $X$ .

Now we will give our main theorem.

Theorem 3. Let  $X$  be the attractor in Theorem 2. Then the Hausdorff dimension  $d_H(X)$  is finite and the following estimate holds :

$$(10) \quad d_H(X) \leq 1 + 2(\gamma_2/\gamma_1 + \sqrt{\gamma_3/\gamma_1}) ,$$

where  $\gamma_1 = C_2(\lambda_1 + \lambda_1')/2(1+P)$ ,  $\gamma_2 = (2/P + |R|/2)$ ,  $\gamma_3 = (C_1 + 4)\|\nabla b\|^2$

$$+ 4C_\Omega^2(\|b\| \cdot \|\nabla b\|^3 + (3|R|/P + d^3\|g\|_\infty^2/\nu^2)^2 \cdot |\Omega|) , \quad |R| = \alpha\|g\|_\infty T_0 d^3/\kappa\nu ,$$

$\|g\|_{\infty}^2 = \|g_1\|_{L^{\infty}}^2 + \|g_2\|_{L^{\infty}}^2$ ;  $\lambda_1$  and  $\lambda_1'$  are the smallest eigenvalues of the Stokes operator and  $-\Delta$  with the homogeneous Dirichlet condition, respectively.

Remark 2. The following estimate is known ([4], p118) :  
The function  $b$  given in Lemma 1 (satisfying (5)) also satisfies an estimate of the form

$$(11) \quad \|b\|_{L^2(\Omega)} \leq \|b\|_{H^1(\Omega)} \leq C \varepsilon \exp(4/\varepsilon) \|\beta\|_{H^{1/2}(\partial\Omega)},$$

where  $C$  depends on the domain  $\Omega$  and physical constants.

Remark 3. We will denote the fractal dimension of  $X$  by  $d_F(X)$ . Then we can obtain the estimate

$$d_F(X) \leq 2 + 4(\gamma_2/\gamma_1 + \sqrt{\gamma_3/\gamma_1}).$$

#### §4. Some lemmas.

To prove the theorems, we prepare some lemmas.

Lemma 2. Let  $X$  be a subset of a Hilbert space  $H$  and  $\{S(t)\}_{t \geq 0}$  be a semigroup in  $H$ . Suppose that  $S(t)X = X$  for any  $t > 0$ ,  $S(t)$  is differentiable on  $X$  with the differential  $L(t, u)$  and  $\sup_{u \in X} \|L(t_0, u)\|_{L(H)} < +\infty$  for some  $t_0 > 0$ . Denote the Lyapunov exponents for  $X$  by  $\mu_j$  ( $j \geq 1$ ). If for some  $n \geq 1$ ,  $\mu_1 + \dots + \mu_{n+1} < 0$ , then  $\mu_{n+1} < 0$ ,  $(\mu_1 + \dots + \mu_n)/|\mu_{n+1}| < 1$  and the Hausdorff dimension  $d_H(X)$  is bounded as

$$(12) \quad d_H(X) \leq n + \frac{(\mu_1 + \dots + \mu_n)_+}{|\mu_{n+1}|} < n+1.$$

The next elementary lemma is also useful ([4], p303).

Lemma 3. We assume that the sequence  $\{\mu_j\}_{j \geq 1}$  satisfies the

following inequalities

$$(13) \quad \mu_1 + \dots + \mu_j \leq -\alpha j^\theta + \beta \quad \text{for any } j \geq 1,$$

where  $\alpha, \beta, \theta > 0$ . Let  $m \in \mathbb{N}$  be defined by

$$(14) \quad m - 1 < (2\beta/\alpha)^{1/\theta} \leq m.$$

Then  $\mu_1 + \dots + \mu_m < 0$  and  $(\mu_1 + \dots + \mu_j)_+ / |\mu_1 + \dots + \mu_m| < 1$

for  $j = 1, \dots, m$ .

To state the next lemma, we prepare a framework as follows.

Let  $\{S(t)\}_{t \geq 0}$  be a semigroup in a Hilbert space  $H$  generated by a nonlinear evolution equation

$$(15) \quad \frac{du}{dt} = F(u(t)) \quad \text{for } t > 0, \quad u(0) = u_0 \in H.$$

We assume (15) has a linearized equation

$$(16) \quad \frac{dU}{dt} = A_F(S(t)u_0)U(t), \quad U(0) = \xi,$$

and moreover we assume (16) is well-posed for any  $u_0$  and  $\xi \in H$ .

Finally we assume  $S(t)$  is differentiable in  $H$  with the differential  $L(t, u_0)$  defined by

$$(17) \quad L(t, u_0)\xi = U(t) \quad \text{for any } \xi \in H,$$

where  $U(t)$  is a solution of (16).

Under these assumptions, we have ([4]) :

Lemma 4. If  $X$  is a functional invariant set of  $S(t)$  and  $\mu_j$  ( $j \geq 1$ ) are Lyapunov exponents for  $X$ , then

$$(18) \quad \mu_1 + \dots + \mu_m \leq q_m,$$

where  $q_m$  is defined by

$$(19) \quad q_m \equiv \limsup_{t \rightarrow \infty} q_m(t)$$



$$= \lim_{t \rightarrow \infty} \sup_{u_0 \in X} \left( \sup_{\substack{\xi_i \in H \\ \|\xi_i\| \leq 1}} \frac{1}{t} \int_0^t \text{Tr} (A_F(u) \circ Q_m(\tau)) d\tau \right)$$

and  $Q_m(t, u_0, \xi_1, \dots, \xi_m)$  is the projector from  $H$  onto the space spanned by  $U_1(t), \dots, U_m(t)$ ;  $U_i(t)$  being solutions of (16) with  $U_i(0) = \xi_i$ .

We use later the known facts as below.

Lemma 5. Let  $\{\lambda_j\}$  and  $\{\lambda_j'\}$  be eigenvalues of the Stokes operator and  $-\Delta$  with the homogeneous Dirichlet condition on  $\Omega$ , respectively. If  $\Omega \subset \mathbb{R}^2$ , then

$$(20) \quad \lambda_j \sim c\lambda_1 j \text{ as } j \rightarrow \infty \text{ (by Metivier),}$$

$$(21) \quad \lambda_j' \sim c\lambda_1' j \text{ as } j \rightarrow \infty \text{ (by Courant-Hilbert).}$$

Lemma 6. ([4].) Let  $A$  be a linear positive self-adjoint operator in a Hilbert space  $H$ . Suppose  $A^{-1}$  is compact. Let  $\{\lambda_j\}$  be eigenvalues of  $A$ . Then, for any family of elements  $\varphi_1, \dots, \varphi_m$  of  $V = D(A^{1/2})$  which is orthonormal in  $H$ ,

$$(22) \quad \sum_{j=1}^m (A\varphi_j, \varphi_j) \geq \lambda_1 + \dots + \lambda_m.$$

If, furthermore,  $\lambda_j \sim c\lambda_1 j^\alpha (\alpha > 0)$  as  $j \rightarrow \infty$ ,  $c$  depending on  $A$ , then

$$(23) \quad \sum_{j=1}^m (A\varphi_j, \varphi_j) \geq \lambda_1 + \dots + \lambda_m \geq c' \lambda_1 m^{\alpha+1},$$

with another constant  $c'$  depending  $A$  and  $\alpha$ .

## §5. Proofs of the results.

We will only give the proof of Theorem 3 which is the main theorem of our work. First, we introduce the linearized equation of (AHC). Let  $\varphi = {}^t(u, \theta)$  be a solution of (AHC) with  $\varphi(0) = \varphi_0$

$= {}^t(u_0, \theta_0)$ . For  $\Phi = {}^t(U, \theta) \in (H^2(\Omega) \cap H_\sigma^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$ ,

we define an operator

$$(24) \quad A_F(\varphi)\Phi = \begin{pmatrix} P_\sigma(\Omega) \{ \Delta U - ((u+b) \cdot \nabla)U - (U \cdot \nabla)(u+b) - R\theta \} \\ (1/P)\Delta\theta - ((u+b) \cdot \nabla)\theta - (U \cdot \nabla)(\theta + \bar{\theta}) \end{pmatrix}$$

Then the linearized equation (LAHC) of (AHC) is given by

$$(LAHC) \quad \frac{d\Phi}{dt} = A_F(\varphi)\Phi, \quad \Phi(0) = {}^t(\xi, \eta).$$

Remark 4. (LAHC) is well-posed for any  ${}^t(\xi, \eta) \in H = H_\sigma(\Omega) \times L^2(\Omega)$ . On the other hand, we can show  $S(t)$  is differentiable in  $H$  and its differential  $L(t, \varphi_0)$  is written for every  ${}^t(\xi, \eta) \in H$  as  $L(t, \varphi_0) {}^t(\xi, \eta) = {}^t(U(t), \theta(t))$  where  ${}^t(U(t), \theta(t))$  is a solution of (LAHC) with  ${}^t(U(0), \theta(0)) = {}^t(\xi, \eta)$ . Therefore, we can apply Lemma 4 to (AHC). Moreover, we see that for some  $t_0 > 0$ ,  $\sup_{\varphi_0 \in X} \|L(t_0, \varphi_0)\| < +\infty$  where  $X$  is the attractor in Theorem 2. whence Lemma 2 is applicable to (AHC). We omit these verification.

Now, let  $X$  be the attractor for (AHC), we define  $q_m$  by

$$(25) \quad q_m = \lim_{t \rightarrow \infty} \sup \left( \sup_{\substack{({}^t u_0 \\ {}^t \theta_0)} \in X} \sup_{\substack{{}^t(\xi_i, \eta_i) \in H \\ \|{}^t(\xi_i, \eta_i)\| \leq 1}} \frac{1}{t} \int_0^t \text{Tr}(A_F(\varphi(\tau)) \cdot Q_m(\tau)) d\tau \right),$$

where  $\varphi = {}^t(u, \theta)$  is a solution of (AHC) with  $\varphi(0) = {}^t(u_0, \theta_0)$  and  $Q_m H$  is spanned by  ${}^t(U_1, \theta_1), \dots, {}^t(U_m, \theta_m)$ ;  ${}^t(U_i, \theta_i)$  are solutions of (LAHC) with  ${}^t(U_i(0), \theta_i(0)) = {}^t(\xi_i, \eta_i)$ . Then we present the following lemma by which we can prove Theorem 3.

Lemma 7. Consider (AHC) equation. Then we have

$$(26) \quad q_m \leq -\gamma_1 m^2 + \gamma_2 m + \gamma_3 \leq -\frac{\gamma_1}{2} m^2 + \frac{\gamma_2^2}{2\gamma_1} + \gamma_3,$$

where  $\gamma_i$  are defined in Theorem 3.

Remark 5. If Lemma 7 is proved, then from (18) of Lemma 4, we get an inequality like a type of (13) of Lemma 3, from which we can find  $m$  such that  $\mu_1 + \dots + \mu_m < 0$  and, using Lemma 2, we conclude that (10) of Theorem 3 holds.

Proof of Lemma 7.

We recall that  $X$  is the attractor for (AHC),  $q_m$  is defined by (25) and  $Q_m$  is the projector. To estimate  $q_m$ , let  $\Psi_j(s) = {}^t(w_j(s), \theta_j(s))$  be an orthonormal basis of  $H$ ,  $\Psi_j \in V$  and  $\Psi_1, \dots, \Psi_m$  span  $Q_m H$ . Now we calculate

$$\begin{aligned} (27) \quad T_r(A_F(\varphi) \circ Q_m) &= \sum_{j=1}^m (A_F(\varphi) \Psi_j, \Psi_j) \\ &= \sum_{j=1}^m \{ (\Delta w_j, w_j) - (((u+b) \cdot \nabla) w_j, w_j) - ((w_j \cdot \nabla)(u+b), w_j) - (R\theta_j, w_j) \} \\ &\quad + \sum_{j=1}^m \{ (\frac{1}{P} \Delta \theta_j, \theta_j) - (((u+b) \cdot \nabla) \theta_j, \theta_j) - ((w_j \cdot \nabla)(\theta + \bar{\theta}), \theta_j) \}. \end{aligned}$$

Here we notice :

$$\begin{aligned} (28) \quad \sum_{j=1}^m |((w_j \cdot \nabla)(u+b), w_j)| &\leq \int_{\Omega} (\sum_{j=1}^m |w_j|^2) \cdot |\nabla(u+b)| dx \\ &\leq \|\rho\| \cdot \|\nabla(u+b)\| \quad (\text{where } \rho(x) \equiv \sum_{j=1}^m |w_j(x)|^2) \\ &\leq C_1 (\|\nabla u\|^2 + \|\nabla b\|^2) + \frac{1}{2} \sum_{j=1}^m \|\nabla w_j\|^2, \end{aligned}$$

here we used  $\|\rho\|^2 \leq C_1 \sum_{j=1}^m \|\nabla w_j\|^2$ ,  $C_1$  depending on  $\Omega$ .

$$(29) \quad \sum_{j=1}^m |(R\theta_j, w_j)| \leq \frac{|R|}{2} m,$$

since  $\|w_j\|^2 + \|\theta_j\|^2 = 1$  (normalized).

$$\begin{aligned} (30) \quad \sum_{j=1}^m |((w_j \cdot \nabla)(\theta + \bar{\theta}), \theta_j)| &= \sum_{j=1}^m |((w_j \cdot \nabla) \theta_j, (\theta + \bar{\theta}))| \\ &\leq \frac{2}{P} \sum_{j=1}^m \|w_j\| \cdot \|\theta_j\| \leq \frac{1}{2P} \sum_{j=1}^m \|\nabla \theta_j\|^2 + \frac{2}{P} m, \end{aligned}$$

where we employed  ${}^t(w_j, \theta_j) \in V$ ,  $\|w_j\| \leq 1$  together with  $|\theta(\cdot, t)|$

$\leq 1/P$  for  $\theta \in x$  and  $|\bar{\theta}| \leq 1/P$  (maximal principle).

Using (28), (29), (30) and noticing

$$(31) \quad \|\nabla w_j\|^2 + P^{-1} \|\nabla \theta_j\|^2 \geq (1+P)^{-1} (\|\nabla w_j\|^2 + \|\nabla \theta_j\|^2),$$

and with the aid of Lemma 5 and Lemma 6, then we have

$$(32) \quad T_r(A_F(\varphi) \circ Q_m) \\ \leq -\frac{1}{2(1+P)} \sum_{j=1}^m (\|\nabla w_j\|^2 + \|\nabla \theta_j\|^2) + \left(\frac{2}{P} + \frac{|R|}{2}\right)m + C_1 (\|\nabla u\|^2 + \|\nabla b\|^2) \\ \leq -\frac{C_2}{2(1+P)} (\lambda_1 + \lambda_1') m^2 + \left(\frac{2}{P} + \frac{|R|}{2}\right)m + C_1 (\|\nabla u\|^2 + \|\nabla b\|^2),$$

where  $C_2$  depends on  $\Omega$ .

Next, we estimate  $\|\nabla u(t)\|^2$ . To do this, recall that  ${}^t(u, \theta)$  is a solution of (AHC), then we get

$$(33) \quad \frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \|\nabla u(t)\|^2 \\ = -((u \cdot \nabla)b, u) - ((b \cdot \nabla)b, u) - (\Delta b, u) - (R\theta, u) \\ + d^3 v^{-2} (g, u) - (R\bar{\theta}, u) + P^{-1} (R, u) \\ \leq 4 \times \frac{1}{8} \|\nabla u\|^2 + 2C_\Omega^2 (\|b\| \cdot \|\nabla b\|^3 + (3|R|P^{-1} + d^3 v^{-2} \|g\|_\infty)^2 |\Omega|) + 2\|\nabla b\|^2,$$

where we used Lemma 1 with  $\varepsilon = 1/8$ .

Thus we obtain

$$(34) \quad \lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t \|\nabla u(\tau)\|^2 d\tau \\ \leq 4\|\nabla b\|^2 + 4C_\Omega^2 (\|b\| \cdot \|\nabla b\|^3 + (3|R|P^{-1} + d^3 v^{-2} \|g\|_\infty)^2 |\Omega|).$$

Hence, finally we have

$$(35) \quad q_m = \lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t T_r(A_F(\varphi) \circ Q_m) d\tau \\ \leq -C_2 2^{-1} (1+P)^{-1} (\lambda_1 + \lambda_1') m^2 + (2P^{-1} + |R| \cdot 2^{-1})m$$

$$+ (C_1 + 4) \|\nabla b\|^2 + 4C_\Omega^2 (\|b\| \cdot \|\nabla b\|^3 + (3|R|P^{-1} + d^3\nu^{-2} \|g\|_\infty)^2 |\Omega|).$$

$$\equiv -\gamma_1 m^2 + \gamma_2 m + \gamma_3 \dots$$

Recalling Remark 4 and 5, we have proved Theorem 3.

#### References.

- [1] Foias, C., Manley, O., and R. Temam, Attractors for the Bénard problem : Existence and physical bounds on their fractal dimension, *Nonlinear Anal.T.M.A.*, 11, 939-967(1987).
- [2] Ōeda, K., Weak and strong solutions of the heat convection equations in regions with moving boundaries, *J.Fac.Sci.Univ. Tokyo, Sect. 1A*, 36, 491-536(1989).
- [3] Ōeda, K., On absorbing sets for evolution equations in fluid mechanics, *RIMS Kokyuroku NO.745*, 144-156(1991).
- [4] Temam, R., *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, 1988.